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A note on the mean correcting martingale measure for geometric Lévy processes[☆]

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ABSTRACT

A martingale measure is constructed by using a mean correcting transform for the geometric Lévy processes model. It is shown that this measure is the mean correcting martingale measure if and only if, in the Lévy process, there exists a continuous Gaussian part. Although this measure cannot be equivalent to a physical probability for a pure jump Lévy process, we show that a European call option price under this measure is still arbitrage free.

1. Introduction

Let r be riskless interest rate and S_t be stock price. We assume $S = (S_t)_{t\in[0,T]}, T > 0$, to be a geometric Lévy process defined on a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ which satisfies the usual conditions. That is, (S_t) is a stochastic process of the following form:

$$S_t = S_0 e^{X_t}, \quad 0 \le t \le T, \tag{1.1}$$

where $S_0 > 0$ is a constant, and $X = (X_t)_{t\in[0,T]}$ is a one-dimensional Lévy process and $X_0 = 0$.

Let $D = D([0, T], R)$ be the space of mappings ξ from $[0, T]$ into R right-continuous with left limits. Write $x_t(\xi) = \xi_t$. Let $\mathcal{F}^D$, $\mathcal{F}_t^D$ be the smallest σ-algebras that make $x_t, t \in [0, T]$, and $x_s, s \in [0, t]$, measurable, respectively. Any Lévy process $(\{X_t\}, P)$ can induce a probability measure P^D on $(D, \mathcal{F}^D)$ such that $(\{x_t\}, P^D)$ is a Lévy process identical in law with $(\{X_t\}, P)$. So we can assume that $\Omega = D$, and $\mathcal{F} = \mathcal{F}^D$. Let X_t be a Lévy process with generating triplets $(\alpha, \sigma^2, \nu)_P$ under P. If the Laplace transform of X_t exists, then we have

$$E^P[\exp(uX_t)] = \exp(t\phi(u)),$$

where $E^P[\cdot]$ denotes the expectation under probability P, and ϕ is given by

$$\phi(u) = u\alpha + \frac{1}{2}\sigma^2 u^2 + \int_R (e^{ux} - 1 - uxI_{|x|<1})\nu(dx). \tag{1.2}$$

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Let $(\triangle X_t)$ be the point process on $R \setminus \{0\}$, where $\triangle X_t = X_t - X_{t-}$. We denote by $N(\mathrm{d}t, \mathrm{d}x)$ the counting measure of the point process $(\triangle X_t)$:

$$N(t, A) = \sharp\{0 \le s \le t : \triangle X_s \in A \text{ for } A \in \mathcal{B}(R \setminus \{0\})\}.$$

Then $N(\mathrm{d}t, \mathrm{d}x)$ is a Poisson measure on $R^+ \times R \setminus \{0\}$ with expectation measure $\mathrm{d}t \times \nu$, where $\mathrm{d}t$ denotes the Lebesgue measure. The Lévy–Ito decomposition says that X_t has the following representation:

$$X_t = \alpha t + \sigma W_t + \int_{(0,t]} \int_{\{|x|>1\}} x N(\mathrm{d}s, \mathrm{d}x) + \int_{(0,t]} \int_{\{|x|\le 1\}} x[N(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}s\nu(\mathrm{d}x)],$$

where (W_t) is a one-dimensional standard Brownian motion.

It is well-known that the model (1.1) is incomplete, so there are many equivalent martingale measures (EMM) in the market. Although the Esscher transform [1] is sometimes easy to obtain, it is not clear that in reality the market chooses this kind of (exponential) transform. Another way to obtain an EMM is by mean correcting the exponential of a Lévy process (see e.g. [2]), which is based on the idea of adjusting the location parameter of a distribution such that a certain drift condition is met. Specifically, for $m \in R$, let $Y_t = X_t - mt$, and Q^m be the probability measure on $(D, \mathcal{F}^D)$ generated by the family of finite dimensional distributions of Y_t; then we have

$$Q^m(Y_t \le x) = P(X_t \le x), \quad x \in R.$$

Under Q^m, X_t is a Lévy process with generating triplets $(\alpha + m, \sigma^2, \nu)_{Q^m}$. Note that this transform only affects the drift of X_t, and it does not influence the infinite divisibility property nor the self-decomposability of the distribution.

We want to ensure that the stock price of the model is internally consistent. Thus we seek $m = m_0$ such that the discounted stock price process $(\mathrm{e}^{-rt}S_t)$ is a martingale with respect to the probability measure corresponding to m_0. For $0 \le s \le t \le T$,

$$\begin{aligned} \mathrm{e}^{-rs}S_s &= E^{Q^{m_0}}[\mathrm{e}^{-rt}S_t|\mathcal{F}_s] \\ &= \mathrm{e}^{-rs}S_s\mathrm{e}^{(t-s)[m_0+\phi(1)-r]}. \end{aligned}$$

We see that the unique solution is

$$m_0 = r - \phi(1).$$

Note that $(\mathrm{e}^{-rt}S_t)$ is a Q^{m_0}-martingale. So Q^{m_0} is an EMM if and only if Q^{m_0} is equivalent to P. If so, Q^{m_0} is called the mean correcting martingale measure (MCMM).

It can be shown that for a pure jump Lévy process, no MCMM can exist. However, it seems that most researchers ignore this fact; they still adopt the "mean correcting martingale measure" as a pricing measure. This work gives us a necessary and sufficient condition for Q^{m_0} to be a MCMM. For a pure jump Lévy process, the corresponding Q^{m_0} cannot be an EMM. It is shown that there must exist an EMM Q' such that the European call option prices under Q' and Q^{m_0} are identical.

2. Main results

In this section, we first discuss the conditions which make Q^{m_0} an EMM. We need the following lemma.

Lemma 2.1. *Let P be a probability measure. Let X be a Lévy process on R^n with generating triplets $(\alpha, \sigma^2, \nu)_P$. Then there is a probability measure $Q \sim P$ such that X is a Q-Lévy process with generating triplets $(\bar{\alpha}, \bar{\sigma^2}, \bar{\nu})_Q$ if and only if there exist $\beta \in R^n$ and a function y from $supp(\nu) \subseteq R^n$ into R_+ satisfying*

$$\int_{R^n} |h(x)(1 - y(x))|\nu(\mathrm{d}x) + \int_{R^n} (1 - \sqrt{y(x)})^2 \nu(\mathrm{d}x) < +\infty$$

and

$$\begin{aligned} &\bar{\alpha} = \alpha + \beta\sigma^2 + \int_{R^n} h(x)(y(x) - 1)\nu(\mathrm{d}x) \\ &\bar{\sigma^2} = \sigma^2 \\ &\frac{\mathrm{d}\bar{\nu}}{\mathrm{d}\nu} = y(x). \end{aligned}$$

What is more, if $Q \sim P$, then the Radon–Nikodym derivative Z of Q with respect to P has the following form:

$$Z_t = \left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \varepsilon(\xi_t), \quad t \in [0, T], \tag{2.1}$$

where $\xi = \beta W + (y(x) - 1) * (N - \nu)$.

For a proof see [3].

Theorem 2.1. *Suppose $\phi(1) < +\infty$. Q^{m_0} is the MCMM if and only if $\sigma > 0$. If Q^{m_0} is the MCMM, then the Radon–Nikodym derivative is*

$$\left.\frac{\mathrm{d}Q^{m_0}}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \frac{\exp(\beta W_t)}{E^P \exp(\beta W_t)}, \tag{2.2}$$

where $\beta = \frac{r-\phi(1)}{\sigma}$.

Proof. X_t is a Lévy process under P and Q^{m_0} with generating triplets $(\gamma, \sigma^2, \nu)_P$ and $(\gamma + (r - \phi(1)), \sigma^2, \nu)_Q$, respectively. It follows from Lemma 2.1 that

$$y(x) \equiv 1$$

and β satisfies the following equation:

$$\sigma\beta = r - \phi(1),$$

from which we conclude that $\beta = \frac{r-\phi(1)}{\sigma}$ if and only if $\sigma > 0$. □

Theorem 2.1 tells us that if $\sigma > 0$, Q^{m_0} is just a MCMM. So the call option price, $E^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T - a)^+]$, of a European call option with time to maturity T and strike price a is an arbitrage free price. However, if $\sigma = 0$, Q^{m_0} cannot be equivalent to P, and then the price $E^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T - a)^+]$ may not be arbitrage free. From now on, we assume $\sigma = 0$, i.e., X is a Lévy process with generating triplets $(\alpha, 0, \nu)_P$.

Let $\mathcal{M}_r$ be the class of measures locally equivalent to P under which $\mathrm{e}^{-rt}S_t$ is a martingale, and $\mathcal{M}'_r$ be the subclass of all $Q \in \mathcal{M}_r$ under which X is also a Lévy process. Under the measure $Q \in \mathcal{M}_r$ the value of the option is then

$$\gamma(Q) = E^Q[\mathrm{e}^{-rT}(S_T - a)^+].$$

Write $I_r = \{\gamma(Q) | Q \in \mathcal{M}_r\}$ and $I'_r = \{\gamma(Q) | Q \in \mathcal{M}'_r\}$.

Lemma 2.2. *Assume that the Lévy measure ν of the Lévy process X under P has the following properties:*

(i) $\nu((-\infty, b]) > 0$ *for all* $b \in R$.
(ii) ν *has no atom and satisfies* $\int_{[-1,0)} |x|\nu(\mathrm{d}x) = \int_{(0,1]} x\nu(\mathrm{d}x) = +\infty$.

Then $\mathcal{M}_r$ is not empty, I_r is the full interval $((S_0 - a\mathrm{e}^{-rT})^+, S_0)$ and I'_r is dense in this interval.

For a proof see [4].

Theorem 2.2. *If $\sigma = 0$, and the Lévy measure ν under P has the above properties* (i), (ii), *then there exists an EMM $Q' \in \mathcal{M}_r$ such that*

$$\gamma(Q') = E^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T - a)^+].$$

Proof. By Lemma 2.2, we only need prove

$$E^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T - a)^+] \in ((S_0 - a\mathrm{e}^{-rT})^+, S_0).$$

First we have $(S_T - a)^+ < S_T$, so

$$E^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T - a)^+] < E^{Q^{m_0}}[\mathrm{e}^{-rT}S_T] = S_0. \tag{2.3}$$

Second we observe that $M_t = \mathrm{e}^{r(T-t)}S_t$ is a Q^{m_0}-martingale, and the function $g(x) = (x - a)^+$, $a > 0$ is convex; we conclude that $(M_t - a)^+$ is a Q^{m_0}-submartingale, so

$$\begin{aligned} E^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T - a)^+] &= \mathrm{e}^{-rT}E^{Q^{m_0}}[(M_T - a)^+] \\ &\geq \mathrm{e}^{-rT}g(\mathrm{e}^{rT}S_0) \\ &= (S_0 - a\mathrm{e}^{-rT})^+. \end{aligned} \tag{2.4}$$

In view of (2.3) and (2.4), it remains to show that the left endpoint $\mathrm{e}^{-rT}g(\mathrm{e}^{rT}S_0)$ does not belong to $E^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T - a)^+]$. Suppose that $E^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T - a)^+] = \mathrm{e}^{-rT}g(\mathrm{e}^{rT}S_0)$. This means that if $g'(x) = g(\mathrm{e}^{rT}x)$ and $U = \mathrm{e}^{-rT}S_T$, then $E^{Q^{m_0}}[g'(U)] = g'(E^{Q^{m_0}}[U])$. Hence the convex function g' should be linear on the interval (α, β), where α and β are the left and right endpoints of the support of the random variable U. Note that the Lévy measure ν does not change after measure transformation from P to Q^{m_0}. Now, the Lévy measure ν charges both R_- and R_+ under Q^{m_0}. Hence the support of X_T extends from $-\infty$ to $+\infty$, and the support of $U = S_0\mathrm{e}^{-rT+X_T}$ extends from 0 to $+\infty$ under Q^{m_0}, i.e. $\alpha = 0$ and $\beta = +\infty$. That is, g' and g must be linear on R_+, which contradicts the convexity of g' and g. □

Theorem 2.2 implies that, although Q^{m_0} is not an EMM for a pure jump Lévy process, the price $E^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T-a)^+]$ is still arbitrage free. So if we only pay close attention to the option price, there is no harm in substituting Q^{m_0} for Q'. In fact, the price of a European call option is given by

$$\mathrm{E}^{Q^{m_0}}[\mathrm{e}^{-rT}(S_T-a)^+] = S_0\mathrm{e}^{-T\phi(1)}\int_y^\infty \mathrm{e}^x F^P_{X_T}(\mathrm{d}x) - a\mathrm{e}^{-rT}(1-F^P_{X_T}(y)), \tag{2.5}$$

where $F^P_{X_T}$ is the distribution of X_T under probability P, and $y=(\phi(1)-r)T-\ln\frac{S_0}{a}$.

3. Some examples

Example 3.1. (Merton's Jump-Diffusion Model)

Merton considered the following jump-diffusion model:

$$S_t = S_0\exp\left\{\mu t+\sigma W_t+\sum_{i=1}^{N_t}Y_i\right\},$$

where W_t is a standard Wiener process, N_t is a Poisson process with intensity λ independent from W and $Y_i\sim N(\alpha,\delta^2)$ are i.i.d. random variables independent from W, N. Let $f(x)$ denote the density function of Y_i. If we define the new measure ν such that for all $A\in\mathcal{B}(R)$, $\nu(A)=\lambda f(A)$, then X_t is a Lévy process with generating triplets $(\mu,\sigma^2,\nu)_P$. By (1.2), we have

$$\phi(1)=\mu+\frac{1}{2}\sigma^2+\lambda\left[\exp\left(\alpha+\frac{1}{2}\delta^2\right)-1\right].$$

From (2.2) we can derive the MCMM Q^{m_0}:

$$\left.\frac{\mathrm{d}Q^{m_0}}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \exp\left\{\frac{r-\phi(1)}{\sigma}W_t-\frac{1}{2}\left(\frac{r-\phi(1)}{\sigma}\right)^2 t\right\}.$$

Under Q^{m_0}, the jump-diffusion model becomes

$$S_t = S_0\exp\left\{\mu^{Q^{m_0}}t+\sigma W_t^{Q^{m_0}}+\sum_{i=1}^{N_t}Y_i\right\},$$

where $W_t^{Q^{m_0}}$ is a standard Wiener process, N_t, Y_i are as above, independent from $W_t^{Q^{m_0}}$, and the drift $\mu^{Q^{m_0}}$ of X_t under Q^{m_0} is

$$\mu^{Q^{m_0}}=r-\frac{1}{2}\sigma^2-\lambda\left[\exp\left(\alpha+\frac{1}{2}\delta^2\right)-1\right].$$

Example 3.2 (*Normal Inverse Gaussian Process*)**.** The normal inverse Gaussian (NIG) distribution with parameters $\alpha>0$, $|\beta|<\alpha$, $\delta>0$, $\mu\in R$, NIG$(\alpha,\beta,\delta,\mu)$ has a Laplace transform given by

$$E^P[\exp(uX_t)]=\exp\left(t\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+u)^2}\right)+\mu tu\right),$$

which implies that

$$\phi(1)=\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+1)^2}\right)+\mu.$$

In fact, the density of the NIG distribution can be given explicitly:

$$f_{\mathrm{NIG}}(x;\alpha,\beta,\delta,\mu)=\frac{\alpha\delta K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\pi\sqrt{\delta^2+(x-\mu)^2}}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right),$$

where $K_1(x)$ denotes the modified Bessel function of the third kind with index $\lambda=1$. Below is a case study based on Shanghai composite index in the period 21 August 2009–20 October 2010. On 20 October 2010, the Shanghai composite index closed at 3030. The annual interest rate r is set to 1.98%. Parameter values may be found in Table 1 and calculated prices can be found in Table 2. We calculate call option prices on the basis of formula (2.5). In Table 2, the arbitrage free price span given by $((S_0-\mathrm{e}^{-rTa})+,S_0)$, which was found by Eberlein and Jacod [4], is also shown. It is obvious that all the prices lie in this interval.

Table 1
Parameter estimation for the NIG model.

Parameters	α	β	δ	μ
Estimate	101.3675	−29.3524	0.0215	0.0066

Table 2
European call option prices for the NIG model.

Time to maturity	a	NIG price	Arbitrage free price span
$T = 1$	3200	12.35913	[0.00, 3030]
$T = 1$	3300	0.14536	[0.00, 3030]
$T = 1$	3400	0.00184	[0.00, 3030]
$T = 2$	3200	10.16682	[0.00, 3030]
$T = 2$	3300	6.609226	[0.00, 3030]
$T = 2$	3400	0.083417	[0.00, 3030]
$T = 3$	3200	61.61117	[14.54, 3030]
$T = 3$	3300	34.26837	[0.00, 3030]
$T = 3$	3400	3.894715	[0.00, 3030]
$T = 5$	3200	228.5554	[131.62, 3030]
$T = 5$	3300	136.4528	[41.05, 3030]
$T = 5$	3400	26.81809	[0.00, 3030]

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